SYSTEMS AND METHODS FOR MODELING CREDIT RISKS OF PUBLICLY TRADED COMPANIES

5 **Field of the Invention**

The present invention relates to methods and systems for modeling risk and more particularly to methods and systems for modeling credit risks of publicly traded companies.

Background of the Invention

10 Evaluating and managing credit risks with respect to publicly traded companies are important capabilities for financial service providers. Credit risk analysis is used to evaluate a multitude of daily-occurring financial transactions including, for example: the origination of bank loans, the buying and selling of bonds, transactions in the credit derivatives market, the extension of payment to customers, portfolio development and management and portfolio valuation.

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The use of Black-Scholes-based analysis tools to estimate credit risk is known in the art. The well known Black-Scholes algorithm uses as inputs values relating to a particular publicly-traded stock, values relating to the financial market as whole and values relating to a proposed stock option in order to calculate a strike price for an option on the particular stock.

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With particular changes to the input variables and the formula of the Black-Scholes algorithm, Robert Merton proposed one of the earliest and simplest default credit models. Merton's model regards the equity in a company as an option on the assets with a strike price equal to the face value of the debt. Merton assumed that the value of the company is governed by a

25 risk-neutralized lognormal stochastic differential equation (SDE) of the form:

$$\frac{dV}{V} = rdt + \sigma dW \qquad (1)$$

Here V is the value of the company, r is the risk neutral rate, σ is the company's volatility, and W is the standard Wiener process, the well-know building block continuous-time stochastic process for constructing random models over continuous time wherein: $W(t)$ for

30 $t \geq 0$, with , $W(0) = 0$ and such that the increment $W(t) - W(s)$ is Gaussian with mean 0 and variance $t - s$ for any $0 \leq s < t$, and increments for non-overlapping time intervals are

independent. Merton chose (somewhat arbitrarily) some maturity *T* and assumed that the company defaults provided that, at maturity, the company value is below the level of its debt *D*. In the case of no default, at maturity equity holders receive the excess amount of *S=V-D*, while debt holders receive the debt amount of *D*. In the case of default, equity holders receive

5   nothing, while debt holders receive the amount of *V*. Thus, Merton views equity as the call option on the value of the company struck at its debt level, and debt as straight bond minus the put option on its value struck at the same level. As a result, the value of the company is the sum of its equity and debt.

10   The actual calculations in the Merton model are straightforward. For finding the value of equity and debt, we can use the usual Black-Scholes formulas:

$$S = VN(d_+) - e^{-rT} DN(d_-) \tag{2}$$

15     $$\delta = VN(-d_+) + e^{-rT} DN(d_-) \tag{3}$$

$$d_\pm = \left[\ln(V/D) + \left(r \pm \sigma^2/2\right)T\right]/\sigma\sqrt{T} \tag{4}$$

wherein:

20   *V*: Value of the company

D : Debt at maturity.

r : Risk free interest rate.

σ: Standard deviation of the underlying asset, eg value of the company.

t Intermediate time between today *(t=0)* and maturity *(t=T)*.

25   *T* : Maturity date.

S: Stock price, or equity

δ: Current debt value

Survival probability $Q(t)$ is defined only for $t=T$:

$$Q(T) = N(d_-) \qquad\qquad (5)$$

5   Here $N(.)$ is the cumulative normal distribution. Equity option with maturity $0 < T_1 < T$ and strike K can be priced as a compound option, and the corresponding equity volatility σ_S be determined by inverting the Black-Scholes formula. As a rule, Merton's model is calibrated to the market by choosing V and σ in such a way that the corresponding S and σ_S are matched exactly.

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Using Merton's model in equation (1) above with the equity and debt values calculated using the Black-Scholes formulae (2), (3), (4) and (5), the probability of default is estimated from the market value of the equity, the face value of the debt and the volatility of the equity. The model is thus used to calculate an estimate of the likelihood of financial failure of a publicly

15   traded company that is whether the debt position of a company will overrun its equity position, resulting in a financial failure.

Several commercial providers, including, for example, Moody-KMV, use Merton's model. Merton's model has obvious attractions because of its simplicity and intuitive character, but it

20   does not address a number of very important questions, for instance, how to price bonds with different maturities in a consistent fashion. Merton's model further suffers from some fundamental mathematical inaccuracies relating to the default event. In particular, Merton's model inherently contemplates small changes, or 'random walks' in values from a starting point to a finishing point, i.e. from the current financial position of a publicly traded stock to a

25   subsequent financial failure. The algorithm does not accommodate the large, random jumps often experienced in the marketplace. Further, the algorithm as initially set out contemplated "European style" option analysis, wherein options were exercised on a specified forward date. Merton's model does not contemplate a default before the due date. In practice, actions associated with financial failure can occur as a result of a single large event or a random walk

30   on any given day. The reader is directed to the Internet websites http://www.moodyskmv.com/about/index.html, and http://www.moodyskmv.com/products/index.html,

for publicly available information on the KMV credit risk analysis products and services.

At a later date, H.E. Leland modified Morton's model to restructure the description of a default event, thereby contemplating earlier default dates than the due date of the debt. Leland's

5   extension of Merton's model assumes that the value of the company is governed by the standard risk-neutralized log-normal SDE, but that default happens whenever the value of the company falls below a certain continuously monitored constant barrier level LD, where L is the relative level of company's liabilities, or at maturity T when the value of the company falls below D. Leland uses the standard formulas for barrier options on lognormal assets in order to

10   find the value of equity S, debt δ, and survival probabilities $Q(t)$, $Q(T)$:

$$S = VN(d_+) - e^{-rT}DN(d_-) - V(LD/V)^{2r/\sigma^2+1}N(f_+) + e^{-rT}D(LD/V)^{2r/\sigma^2-1}N(f_-) \quad (6)$$

$$\delta = VN(-d_+) + e^{-rT}DN(d_-) + V(LD/V)^{2r/\sigma^2+1}N(f_+) - e^{-rT}D(LD/V)^{2r/\sigma^2-1}N(f_-) \quad (7)$$

15    $$f_\pm = \left[\ln(L^2 D/V) + (r \pm \sigma^2/2)T\right]/\sigma\sqrt{T} \quad (8)$$

$$Q(t) = N(g_+) - (LD/V)^{2r/\sigma^2}N(g_-), \quad t < T \quad (9)$$

$$g_\pm = \left[\pm\ln(V/LD) + (r - \sigma^2/2)t\right]/\sigma\sqrt{t} \quad (10)$$

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$$Q(T) = N(d_-) - (LD/V)^{2r/\sigma^2-1}N(f_-) \quad (11)$$

where $d_\pm$ are defined by equation (4) and the remaining variables are as defined above.

25   Leland's model is rich enough for the user to be able to find the *Credit Default Spread* (CDS), i.e., the extra discount value provided for risky bonds as compared to non-risky bonds:

$$CDS(t) = (1 - R)\left(\frac{1 - e^{-rt}Q(t)}{\int_0^t e^{-rt'}Q(t')dt'} - r\right) \quad (12)$$

where R is the recovery level for a particular debt seniority. While for medium and long maturities this spread is broadly comparable with the market, for short maturities it is very low, in sharp contrast with reality.

5   To rectify the above-mentioned problem, Credit Grades, a joint venture between JP Morgan, Deutsche Bank, and Goldman Sacks, proposed to make the recovery level L a normal random variable with the expected value of $\bar{L}$ and volatility ϖ. By doing so, they effectively replaced the calendar time t with the "shifted" time $\varsigma = t + \varpi^2 / \sigma^2$ and hence replaced the expression for the survival probability described above in equation (9) with the following:

10      $$Q(t) = N(\hat{g}_+) - (LD/V)^{2r/\sigma^2} N(\hat{g}_-), \quad t < T \tag{13}$$

$$\hat{g}_{\pm} = \left[\pm \ln(V/LD) + (r - \sigma^2/2)\varsigma \right] / \sigma \sqrt{\varsigma} \tag{14}$$

As a result, they managed to increase the short-term CDSs, which still is given by equation
15   (12). However, the model suffered in that it implicitly assumes that the value of the company can be below its default level from the very beginning, which, in the overwhelming majority of cases, is not true. Also, due to the approximate character of this formula, the initial survival probability $Q(0)$ is less than one, which is mathematically (and financially) impossible. For further publicly available information the reader is directed to the Internet web page
20   http://www.creditgrades.com/intro/intro.

In the later 1990's C.A. Zhou developed a jump-diffusion process, under which a company could default instantaneously because of a sudden drop in its value. The Zhou process was an extension of Leland's model, and addressed the issue of low CDSs in a more direct way. Zhou
25   assumed that the value of the company is governed by an SDE with jumps:

$$\frac{dV}{V} = (r - \lambda\kappa)dt + \sigma dW + (e^J - 1)dN \tag{15}$$

Here N is the standard Poisson process with intensity λ, J is the lognormal jump size, which is assumed to be a random variable with a known probability density function $\phi(J)$, and κ is the expected value of the jump size, $\kappa = \int_{-\infty}^{\infty} (e^J - 1)\phi(J)dJ$. All other values are as defined above.

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As in Leland's framework, default occurs if $V(t)$ crosses the barrier LD, or, alternatively, if $V(T)<D$. The value of the company's equity S and its debt δ are computed by solving pricing problems for the corresponding barrier options. Due to the possibility of the company's value jumping down, CDSs for short maturities are significant. The key drawback of Zhou's model

5   is its computational complexity. Also, its treatment of the recovery is somewhat *ad hoc*. It is very well known that barrier pricing problems for jump diffusions are very complicated. Zhou solves them via a *M*onte *C*arlo (MC) method. While MC methods are versatile, they are inaccurate and slow. Zhou's jump-diffusion model can match the size of credit spreads on corporate bonds and can generate various shapes of yield spread curves and marginal default

10   rate curves, including upward-sloping, downward-sloping, at, and hump-shaped, even if the company is currently in good financial standing. The Zhou model also links recovery rates to company value at default in a natural way so that variation in recovery rates is endogenously generated in the model

15   Despite the improvements to the Merton's model offered by Leland and then Zhou, significant challenges relating to accuracy and operability exist with today's credit risk models. The present inventors believe that even with the Leland and Zhou improvements, the likelihood and timing of default is not accurately modeled. Further, many implementations of the various models are very difficult to program and execute on a computer.

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Commercially available credit risk models exist which attempt to compensate for some of the deficiencies of described above.

The credit underlying securities pricing model, CUSP®, available from Credit Suisse First

25   Boston™, is another commercially available credit risk model. CUSP® is an analytical model that relates an issuer's capital structure, stock price and the option implied volatility of its shares to credit risk. CUSP® provides systematic monitoring of credit risk from forward-looking, market-based measures (One Standard Deviation Spread Widening Risk (SWR)) and relative value tracking which incorporates both risk and return (Probability Weighted Return

30   (PWR)). For further publicly available information, the reader is directed to the Internet website http://www.csfb.com/institutional/research/cusp.shtml.

While useful for estimating credit risks, the present inventors believe that the currently available algorithms and commercial models of the type described above still include inaccuracies in modeling both the random walks and the larger jumps experienced in the marketplace, as noted above. The more accurately such processes can be modeled, the higher the validity of a credit risk analysis tool. Not only is it desirable to accurately model such activities, but also the model itself must be able to be implemented, for example in a spreadsheet or other computer program, so as to practically operable.

Summary of the Invention

In accordance with one aspect of the invention, the present inventors have developed a credit risk model that accurately incorporates the random walk and larger jump events actually experienced in the marketplace.

In accordance with another aspect of the invention, the present inventors have developed a methodology of solving the improved credit risk model that is readily implemented on a computer in spreadsheet or other digital program formats.

There are shown and described new structural default models for modeling the likely default of publicly traded companies. In a first embodiment, the invention is straight-forward to implement and allows the capture of some important ingredients of the actual default, including positive short-term CDSs. This first embodiment can be viewed as a hybrid between a structural and a reduced form model. A second embodiment is somewhat more versatile and complex. It provides a very efficient method, which is by far superior to the Monte Carlo method, for dealing with the term structure of a default boundary, jumps in the company's value, etc.. This second embodiment enables the deep analysis of the term structure of the debt obligations of a given company, and the company's ability to pay back its debt. Further provided is a process using Fast Fourier Transform matrix processing for processing the structural default models in a computationally efficient manner.

In the first embodiment, methods and systems are provided for calculating the financial status of a company, the method comprising the steps of:

calculating the company's value in accordance with the formula

$$\frac{dV}{V} = (r + \lambda)dt + \sigma dW - dN$$

wherein

V is the value of the company,

r is the risk neutral rate,

5   λ is the intensity of jump arrivals,

σ is the company's volatility,

N is the standard Poisson process,

W is the standard Wiener process,

and t is a time between 0 and T, the maturity of the debt,; and

10   using the company value to calculate a financial metric for the company.

In the second embodiment, methods and systems are provided for for calculating the financial status of a company, the method comprising the steps of: calculating the value over time of a company in accordance with Zhou's model; determining that the company defaults if at a

15   sequence of discrete observational times the value of the company falls below a corresponding barrier level, the barrier levels selected to represent different debt amounts which come due at corresponding times; and calculating the transitional probability density function for the value of the company conditional on no default occurring between an initial time and an observational time using a probability vector P.

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Brief Description of the Drawing Figures

These and other objects, features and advantages of the invention will be apparent from a consideration of the following detailed description of the invention when read in conjunction with the drawing figures, in which:

25   Figure 1 is a plot showing survival probability versus time calculated using the closed form solution described below;

Figure 2 is a plot showing credit default spread versus time calculated using the closed form solution described below;

Figure 3 is a plot showing survival probability versus time calculated using the general

30   solution described below; and

Figure 4 is a plot showing credit default spread versus time calculated using the general solution described below.

Detailed Description of the Invention

The present invention is described below with respect to two embodiments. The first embodiment comprises a closed form solution with a jump that would bring a company value to 0. The second embodiment is more general, and can be used to analyze how a particular company can default in great detail. In both cases there are provided new and very efficient methods for solving the corresponding barrier pricing problem, calculating CDSs, etc. A process is then provided for solving the first and second embodiments on a computer.

For purposes of explanation, each equation is numbered with an identifying number to the far right. Further for purposes of explanation, a defined variable retains its initial definition unless it is expressly redefined elsewhere.

Closed Form Solution

As noted above, the closed form solution is a relatively straight-forward solution accommodating a value jump to zero. Specifically, it is assumed that the evolution of a company's value is governed by the SDE of the form:

$$\frac{dV}{V} = (r + \lambda)dt + \sigma dW - dN \tag{16}$$

In contrast to Merton's model equation (1) and Zhou's process equation (15), it is assumed that the value of the company either diffuses gradually, or jumps all the way to zero suddenly. This assumption allows the calculation of the relevant quantities in a straight-forward manner. Since the value of the company does not jump with probability $e^{-\lambda T}$ but jumps to zero with probability $1 - e^{-\lambda T}$, equity S, and survival probability $Q(t)$ are represented in the form:

$$S = VN(d_+) - e^{-(r+\lambda)T}DN(d_-) - V(LD/V)^{2(r+\lambda)/\sigma^2 + 1}N(f_+) + e^{-(r+\lambda)T}D(LD/V)^{2(r+\lambda)/\sigma^2 - 1}N(f_-) \tag{17}$$

$$Q(t) = e^{-\lambda T}\left[N(g_+) - (LD/V)^{2r/\sigma^2}N(g_-)\right] \quad t < T \tag{18}$$

$$d_\pm = \left[\ln(S/D) + (r + \lambda \pm \sigma^2/2)T\right]/\sigma\sqrt{T} \tag{19}$$

$$f_\pm = \left[\ln(L^2 D/V) + (r + \lambda \pm \sigma^2/2)T\right]/\sigma\sqrt{T} \tag{20}$$

$$g_{\pm} = \left[\pm \ln(V / LD) + \left(r + \lambda - \sigma^2 / 2\right)t\right] / \sigma \sqrt{t} \qquad (21)$$

Where:

V is the value of the company,

5   r is the risk neutral rate,

σ is the volatility of company's value,

λ is the is the intensity of jump arrivals

D is the value of the Debt to be paid at time T,

N is the standard Poisson process for determining the probability of an event in a time

10   *interval,*

$N(.)$ is the cumulative normal distribution,

T is the maturity of debt,

L is the relative level of company liabilities,

t is a time between 0 and T, the maturity of the debt,

15    $d_{\pm}$ is a mathematical function defined by formula (19),

$f_{\pm}$ is a mathematical function defined by formula (20),

$g_{\pm}$ is a mathematical function defined by formula (21).

By construction, *CDS(t)*, which is given by formula (12), approaches $\lambda(1 - R)$ when t

approaches zero. Here R is the recovery level, which can be different for different seniorities

20   of the debt.

With reference now to Figures 1 and 2, there is shown a survival probability versus time plot

and a credit default spread versus time plot, respectively, for the closed form solution.

25   The relevant parameters for the plots are V=130, $\lambda = 0.1$, r=0, $\sigma = 0.5$, T=10, D=100, L=0.5,

R=0.5.

As can be seen from a consideration of the plots, the relatively steep slope of the survival

probability plot near time equal to 0 in Figure 1, evidenced by the relatively high initial credit

30   default spread of almost 4% at time equal to 0 in Figure 2, is a relatively real-life result arising

from the jumps introduced by the present invention. In contrast, the result provided by the

prior art such as equations (9) and (12) described above typically yield no initial slope in the survival probability curve and a zero credit default spread near time equal to 0.

General Solution

5   This second embodiment of the invention is significantly more general than the closed form solution described above. In this embodiment it is assumed that the evolution of a company's value is given by equation (15) (set out again immediately below).

$$\frac{dV}{V} = (r - \lambda\kappa)dt + \sigma dW + (e^J - 1)dN \tag{15}$$

To be specific, we choose the lognormal distribution of jump sizes, so that J is a normal

10   variable with mean j and standard deviation k. However, rather than assuming that the company defaults when $V(t)$ crosses a flat barrier LD as Zhou did, it is assumed that the company defaults if at a sequence of discrete observational times

$t_0 = 0(today), t_1, t_2, ..., t_N = T(maturity)$ the value of the company $V_n = V(t_n)$ falls below some

barrier levels $B_1, B_2, ..., B_N = D$. Barrier levels represent different debt amounts, which come

15   due at times $t_1, t_2, ..., t_N = T(maturity)$. This flexibility allows the analysis of companies with different term structures of debt obligations and the analysis of their ability to meet their debt obligations in great detail. This embodiment of the invention allows for volatility, drift and other parameters characterizing the evolution of the value of the company to have a piece-wise constant term structure.

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One key feature of the invention is the analysis that the transitional probability density function (TPDF) for the value of the company conditional on no default occurring between today and observational time t, can be computed very efficiently and quickly via the method of forward induction. There is first chosen the range for the natural logarithm of the value of the

25   company $U=ln(V)$:

$$U_{min} < U < U_{max} \tag{22}$$

where

$$U_{min} = -nstdiv\sqrt{(\sigma^2 + k^2)T} + \min(r - \lambda\kappa - \sigma^2/2,0)T \tag{23}$$

30   $$U_{max} = nstdiv\sqrt{(\sigma^2 + k^2)T} + \max(r - \lambda\kappa - \sigma^2/2,0)T \tag{24}$$

Here *nsd* is the number of standard deviations, used to characterize the extreme values of the natural logarithm of the value of the company, typically chosen *nsd* to be 3 or 4 since the probability of this value to move outside of the chosen interval is negligible because of the well-known properties of the normal distribution. Next this range is divided into an equidistant grid $u_0 = U_{min}, u_1, ..., u_m, ..., u_M = U_{max}$ of M steps, with grid step, denoted by *h,* equal to $h = (U_{max} - U_{min})/M$. For subsequent calculations to perform properly, it is required that the total number of points in the grid, which is equal to *M+1*, be a power of 2, for example, *M=127, M=255, M=511*, etc.

Next, there is defined the integer $\mu = floor(-U_{min}/h)$, where *floor(.)* is a mathematical function, which for every number defines an integer less than or equal to that number. Next there is constructed the modified grid as follows: $\tilde{u}_m = (m - \mu)h, \ m = 0,1,...,M$. Finally, there is defined a grid on the line representing the value of the company via the following formula:

$$v_m = V_0 \exp(u_m) \quad m = 0,1,..., M \ .(24 \ A)$$

The initial value of the company V_0 is equal to V_μ.

On the grid the probability that the company will have a value of v_m at time t_n is given by the vector $P_n = (p_{0,n},..., p_{m,n},..., p_{M,n})$. Initial vector $P_0 = \left(0,...,\underset{\mu}{1/h},...,0\right)$ where the only known zero element is in position mu, To move from the initial probability vector P_0 to the subsequent vector P_1, two operations are performed. First, there is computed the unconditional probability vector $\overline{P}_1$ given by the expression:

$$\overline{P}_1 = \hat{T}_{0,1} P_0 \tag{25}$$

Here $\hat{T}_{0,1}$ is the transition operator between the times t_0 and t_1 for the SDE (see equation 15). Next, there is applied a projection operator $\hat{\Pi}_1$ to vector $\overline{P}_1$ in order to obtain the vector P_1:

$$P_1 = \hat{\Pi}_1 \overline{P}_1 \tag{26}$$

Matrix Multiplication Computational Solution

It will be understood that while the transition operator $\hat{T}_{0,1}$ is rather complicated to efficiently

analyze, the projection operator $\hat{\Pi}_1$ acts on a given vector in a very simple way, that is, it

replaces all the components with indices m, such that $v_m < B_1$, by zeroes. Basic application of

5   the standard rules of matrix multiplication to calculate $\overline{P}_1$ requires $O(M^2)$ operations, where

$O(.)$ is the standard mathematical symbol which means "of order", which would typically

make this operation computationally complex and expensive.

However, as we will be shown, the present invention provides a mathematically elegant

10   shortcut, allowing the calculation of $\overline{P}_1$ by using $O(M \ln(M))$ operations, thus providing

orders of magnitude acceleration in speed in comparison to the standard solution. For instance,

if $M=127$ the method provided by the present invention is approximately 20 times faster than

the standard one.

15   In general, to calculate the company value probability vector P_n it is necessary to perform a

sequence of recursive operations starting with the initial vector P_0:

$$P_1 = \hat{\Pi}_1 \hat{T}_{0,1} P_0,$$

...

$$P_n = \hat{\Pi}_n \hat{T}_{n-1,n} P_{n-1},$$

...

$$P_N = \hat{\Pi}_N \hat{T}_{N-1,N} P_{N-1}, \tag{27}$$

Once all the vectors P_n, $n=0,1,...,N$, are found, survival probabilities can be simply

calculated, and hence CDSs, equity and debt prices, recovery values, etc. by simple

20   substitution of the vectors P_n into the appropriate equations. For example, survival probability

Q_n is calculated as:

$$Q_n = \sum_{m=0}^{M} P_{m,n} \tag{28}$$

where

m is an index changing from 0 to M,

25   M is the total number of grid points,

n is an index changing from 0 to N, and

N is the total number of times when company value is observed.

Equity value S is calculated as:

5

$$S = e^{-rT} \sum_{m=0}^{M} P_{m,N} \max(v_m - B_N, 0) \qquad (29)$$

where

r is the interest rate,

T is the maturity of debt,

10   m is an index changing from 0 to M,

v_m represents the value of the company,

N is the total number of times when company value is observed,

M is the total number of grid points, and

B_N is the terminal debt level.

15

where v_m = one of the possible values of the company at time T and the other values are as defined above.

The present value of company debt δ is calculated as:

20

$$\delta = e^{-rT} \sum_{m=0}^{M} P_{m,N}$$

(29A)

where

r is the interest rate,

T is the maturity of debt,

25   N is the total number of times when company value is observed,

m is an index changing from 0 to $M,$ and

M is the total number of grid points.

and credit default spread CDS is calculated as:

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$$CDS_n = (1-R)\left(\frac{1-e^{-r t_n}Q_n}{\frac{1}{2}e^{-r t_0}(t_1-t_0)+\frac{1}{2}\sum_{n'=1}^{n-1}e^{-r t_{n'}}(t_{n'+1}-t_{n'-1})+\frac{1}{2}e^{-r t_n}(t_n-t_{n-1})}-r\right) \qquad (29B)$$

more particularly, where CDS_n is the credit default spread to time t_n with $1 \leq n \leq N$, and n' is a dummy index,

where

r is the interest rate,

t_n is the observational time between today and maturity T,

R is a recovery level for a selected debt seniority,

n is an index changing from 1 to N, and

n' is an index changing from 1 to n-1.

Matrix elements of a typical projection operator $\hat{\Pi}_n$ are straight-forwardly expressed as

$\rho_{pq} = 1$ if $p = q$ and $v_q \geq B_n$,

$\rho_{pq} = 0$ $otherwise$ $\qquad (30)$

Thus, a vector $y = \hat{\Pi}_n x$ is represented as follows

$$\begin{pmatrix} y_0 \\ \cdots \\ y_m \\ \cdots \\ y_M \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ \cdots \\ x_m \\ \cdots \\ x_M \end{pmatrix} \qquad (31)$$

or, explicitly,

$y_p = x_p$ if $v_q \geq B_n$, $\quad y_p = 0$ $otherwise$ $\qquad (32)$

Accordingly, multiplication by the projection operator is very straight-forward.

The transition operator is more complex. We can represent a typical element $\theta_{p,q}$ in the form:

$$\theta_{p,q} = \theta_{p,q}^{(0)} + \theta_{p,q}^{(1)} \qquad (33)$$

where

$$\theta_{p,q}^{(0)} = \frac{1}{1+\lambda\Delta t}\frac{\exp\left\{-\left(v_q - v_p - (r-\lambda\kappa)\Delta t\right)^2 / 2\sigma^2\Delta t\right\}}{\sqrt{2\pi\sigma^2\Delta t}} \qquad (34)$$

$$\theta_{p,q}^{(1)} = \frac{\lambda\Delta t}{1 + \lambda\Delta t} \frac{\exp\left\{-\left(v_q - v_p - (r - \lambda\kappa)\Delta t - j\right)^2 / 2\left(\sigma^2\Delta t + k^2\right)\right\}}{\sqrt{2\pi\left(\sigma^2\Delta t + k^2\right)}} \qquad (35)$$

Here $\Delta t = t_n - t_{n-1}$. When writing equations (31), (32), we assume that no more that one jump can occur between the observation times, an assumption that is benign and can be dropped if the need occurs. Thus, a vector $y = \hat{T}_{n-1,n}x$ can be represented as follows

$$\begin{pmatrix} y_0 \\ \dots \\ y_m \\ \dots \\ y_M \end{pmatrix} = \begin{pmatrix} \theta_{00} & \dots & \theta_{0m} & \dots & \theta_{0M} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \theta_{mm} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \theta_{M0} & \dots & \theta_{Mm} & \dots & \theta_{MM} \end{pmatrix} \begin{pmatrix} x_0 \\ \dots \\ x_m \\ \dots \\ x_M \end{pmatrix} \qquad (36)$$

It is clear that

$$\theta_{pq} = \theta_{p+1,q+1} \qquad (37)$$

which means that the transition matrix is a Toeplitz matrix, which has identical elements on each diagonal. Every Toeplitz matrix of size $(M+1)x(M+1)$ can be uniquely represented by a vector of size $2(M+1)x1$. For the transition matrix of interest, the corresponding vector, denoted by Θ, has elements of the form:

$$\Theta_0 = \theta_{0,0}, \dots \Theta_m = \theta_{0,m}, \dots, \Theta_M = \theta_{0,M},$$
$$\Theta_{M+1} = 0,$$
$$\Theta_{M+2} = \theta_{M0}, \dots \Theta_{M+m} = \theta_{M-m,0}, \dots, \Theta_{2M+1} = \theta_{1,0} \qquad (38)$$

or, explicitly,

$$\Theta = \left(\theta_{00}, \dots, \theta_{0,m}, \dots, \theta_{0,M}, 0, \theta_{M,0}, \dots, \theta_{M-m,0}, \dots, \theta_{1,0}\right)^T \qquad (39)$$

where the superscript T denotes the transpose. In order to perform the matrix multiplication efficiently, the *Fast Fourier Transform* (FFT) method is applied. This method is well known in the art. For example, Press *et al.* describe this method in great detail. The $(M+1)x1$ probability density vector P is augmented with zeroes in order to increase its length to $2(M+1)x1$, and the extended vector denoted as $\tilde{P}$, so that

$$\tilde{P} = \left(P_0, \dots, P_m, \dots, P_M, 0, \dots, 0 \dots, 0\right)^T \qquad (40)$$

The convolution theorem is applied to show that the vector $\hat{T}P$ is equal to the first $(M+1)$ components of the vector obtained via the following procedure:

$$F^{-1}\left\{F\{\Theta\} * F\{\tilde{P}\}\right\} \qquad (41)$$

where F and F^{-1} denote the direct and inverse FFT, respectively, and * denotes the component-wise multiplication of vectors. It is well-known that an individual FFT requires $O(Mln(M))$ operations, and so does the entire matrix multiplication.

5   It will be appreciated that, in accordance with the above-described matrix computational solution, the above-described processes can be implemented in a straight-forward manner on a conventional computer. For example, a personal computer, workstation or mainframe, using a conventional software program, for example Fortran, C++, or even a conventional spreadsheet such as Microsoft Excel™.

10

With reference now to Figures 3 and 4, there is shown a survival probability versus time plot and a credit default spread versus time plot, respectively, for the general solution.

The relevant parameters are V=130, $\lambda = 0.1$, j=-2, k=0, r=0, $\sigma = 0.5$, T=10, D=100, L=0.5,

15   R=0.5, N=20, nsd=4.

As can be seen from a consideration of the plots, the relatively steep slope of the survival probability plot near time equal to 0 in Figure 3, evidenced by the relatively high initial credit default spread of over 5% near time equal to 0 in Figure 4, is a relatively real-life result arising

20   from the jumps introduced by the present invention. In contrast, as noted above, the result provided by the prior art such as equations (9) and (12) described above typically yield no initial slope in the survival probability curve and a zero credit default spread near time equal to 0.

25   There have thus been shown and described new structural default models for modeling the likely default of publicly traded companies. In a first embodiment, the invention is straight-forward to implement and allows the capture of some important ingredients of the actual default, including positive short-term CDSs. This embodiment can be viewed as a hybrid between a structural and a reduced form model. A second embodiment is somewhat more

30   versatile and complex. It provides a very efficient method, which is by far superior to the Monte Carlo method, for dealing with the term structure of a default boundary, jumps in the company's value, etc.. This second embodiment enables the deep analysis of the term

structure of the debt obligations of a given company, and the company's ability to pay back its debt. Further provided is a process using Fast Fourier Transform matrix processing for processing the structural default models in a computationally efficient manner.

5   While the invention has been shown and described with respect to specific embodiments, it is not thus limited. Numerous changes, modifications and improvements within the scope of the invention will now be apparent to the reader.